COMPLEX ANALYSIS
TOPIC XI: COMPLEX INTEGRATION

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1. Piecewise Smooth Paths

Definition 1. Let $I \subset \mathbb{R}$ be an interval and let $\gamma : I \to \mathbb{C}$.

We say that γ is *continuous* at $t_0 \in I$ if

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \ni \; |t - t_0| < \delta \; \Rightarrow \; |\gamma(t) - \gamma(t_0)| < \epsilon.$$

We say that γ is continuous on I if it is continuous at every point in I.

We say that γ is *differentiable* at $t_0 \in I$ if t_0 is in the interior of I, and there exists $d \in \mathbb{C}$ such that

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \ni \; 0 < |t - t_0| < \delta \; \Rightarrow \; \Big| \frac{\gamma(t) - \gamma(t_0)}{t - t_0} - d \Big| < \epsilon.$$

If such a d exists, we call it the *derivative* of γ, and write $\gamma'(t) = d$. We say that γ is differentiable on I if it is differentiable at every interior point of I.

We say that γ is *nonvanishing* on I if $\gamma(t) \neq 0$ for all $t \in I$.

We note that the definition of continuity works for endpoints, but we insist that differentiability only refers to interior points. It can be shown that if γ is differentiable at t_0, then it is continuous at t_0.

Definition 2. Let $a, b \in \mathbb{R}$ with $a < b$. A *path* in $\mathbb{C}$ is a continuous function $\gamma : [a, b] \to \mathbb{C}$. The *initial point* of γ is $\gamma(a)$, and the *terminal point* of γ is $\gamma(b)$.

We say that γ is *smooth* at t_0 if γ is differentiable on $[a, b]$ with a nonvanishing continuous derivative at t_0. We say that γ is *smooth* if it is smooth at every point in (a, b).

We say that γ is *piecewise smooth* if the there exist $a < t_0 < t_1 < \cdots < t_n = b$ such that γ is smooth on $[t_i, t_{i+1}]$.

A *contour* is the image of a piecewise smooth path, with the initial and terminating points specified.

If γ is a path, the image of γ is often referred to as the *trace* of γ. We say that γ *traces* its contour, from the initial point to the terminating point.

A function $\gamma : I \to \mathbb{C}$ can be written in the form $\gamma(t) = x(t) + iy(t)$, where $x(t) = \mathrm{Re}(\gamma(t))$ and $y(t) = \mathrm{Im}(\gamma(t))$. Clearly $x : [a, b] \to \mathbb{R}$ and $y : [a, b] \to \mathbb{R}$. It can be shown that γ is continuous, differentiable, or smooth, if and only if x and y are continuous, differentiable, and smooth. Moreover,

$$\gamma'(t) = x'(t) + iy'(t).$$

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2. Composition of Piecewise Smooth Paths with Analytic Functions

A situation which often arises involves a function $f : D \to \mathbb{C}$, where D is an open subset of $\mathbb{C}$, and $\gamma : [a,b] \to D$. We wish to study the behavior of f along γ, so we consider the composition $f \circ \gamma : [a,b] \to \mathbb{C}$. We would like to begin by placing some restrictions on f.

We will say that f is *analytic* at a point $z_0 \in D$ if there exists an open neighborhood of z_0 such that f' exists and is continuous at every point in D. We say that f is analytic if it is analytic at every point in its domain. Later, we will show that the premise of continuity can be dropped from this assumption.

Proposition 1 (Chain Rule)**.** *Let D be an open subset of $\mathbb{C}$. Let $f : D \to \mathbb{C}$ be analytic. Let $\gamma : [a,b] \to D$ be piecewise smooth, and suppose that γ is smooth at $t_0 \in [a,b]$. Then $f \circ \gamma$ is smooth at t_0, and*

$$(f \circ \gamma)'(t) = f'(\gamma(t))\gamma'(t).$$

Moreover, $f \circ \gamma$ is piecewise smooth.

3. Path Integrals

Definition 3. Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth path. Write $\gamma(t) = x(t) + iy(t)$, where $x, y : [a,b] \to \mathbb{R}$. The *path integral of* γ is

$$\int_a^b \gamma(t)\,dt = \int_a^b x(t)dt + i\int_a^b y(t)\,dt.$$

Example 1. Compute the integral of $\gamma : [0,\pi] \to \mathbb{C}$ given by $\gamma(t) = e^{it}$.

Solution. Note that $\gamma(t) = \cos t + i \sin t$, so

$$\begin{aligned} \int_0^\pi \gamma(t)\,dt &= \int_0^\pi \cos t\,dt + i\int_0^\pi \sin t\,dt \\ &= [\sin\pi - \sin 0] - i[\cos\pi - \cos 0] \\ &= -0 - i(-1-1) \\ &= 2i. \end{aligned}$$

□

It is clear that path integrals are linear.

Proposition 2. *Let $\alpha, \beta : [a,b] \to \mathbb{C}$ be piecewise smooth. Let $c \in \mathbb{C}$ be constant. Then $\alpha + \beta$ and $c\alpha$ are also piecewise smooth, and*

- **(a)** $\int_a^b \alpha(t) + \beta(t)\,dt = \int_a^b \alpha(t)\,dt + \int_a^b \beta(t)\,dt$;
- **(b)** $\int_a^b c\alpha(t)\,dt = c\int_a^b \alpha(t)\,dt$.

The following fact is also immediate from the definition.

Proposition 3. *Let $\alpha : [a,b] \to \mathbb{C}$ and $\beta : [b,c] \to \mathbb{C}$, with $\alpha(b) = \beta(a)$. Define*

$$\gamma : [a,c] \to \mathbb{C} \text{ by } \gamma(t) = \begin{cases} \alpha(t) & \text{if } t \in [a,b]; \\ \beta(t) & \text{if } t \in [b,c]. \end{cases}$$

Then

$$\int_a^c \gamma(t)\,dt = \int_a^b \alpha(t)\,dt + \int_b^c \beta(t)\,dt.$$

Definition 4. Let D be an open subset of $\mathbb{C}$ and let $f : D \to \mathbb{C}$ be analytic. Let $\gamma : [a, b] \to D$ be a piecewise smooth path. The *path integral of f along γ* is

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

Here, we take the perspective that $z = \gamma(t)$, so $\dfrac{dz}{dt} = \gamma'(t)$, so that $f(z)\,dz = f(\gamma(t))\gamma'(t)\,dt$. Note that under the hypothesis, $f(\gamma(t))\gamma'(t)$ is a piecewise smooth path.

Example 2. Let $f(z) = 1$. Find $\int_\gamma f(z)\,dz$.

Solution. Let $\gamma(t) = x(t) + iy(t)$. We have

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_a^b \gamma'(t)\,dt \\ &= \int_a^b x'(t)\,dt + i\int_a^b y'(t)\,dt \\ &= (x(b) - x(a)) + i(y(b) - y(a)) \\ &= \gamma(b) - \gamma(a).\end{aligned}$$

□

Note that if the path were to go in the opposite direction, then the value of the integral would be negated.

4. Change of Parameter

Path integrals are also called *line integrals*, or *contour integrals*. The latter term comes from the fact that the value of the integral depends only on the contour, and not on its parameterization (although the direction does matter).

Proposition 4. *Let $\phi : [c, d] \to [a, b]$ be a differentiable increasing surjective function of a real variable $s \in [c, d]$. Let $\alpha = \gamma \circ \phi$. Then α has the same trace as γ, and*

$$\int_\alpha f(z)\,dz = \int_\gamma f(z)\,dz.$$

Proof. Since ϕ is increasing, it is invertible with differentiable inverse; indeed, since $\phi(s) = t \in [a, b]$, we can write $\phi'(s) = \dfrac{dt}{ds}$. We have

$$\begin{aligned}\int_\alpha f(z)\,dz &= \int_c^d f(\alpha(s))\alpha'(s)\,ds \\ &= \int_c^d f(\gamma(\phi(s)))\gamma'(\phi(s))\frac{dt}{ds}\,ds \\ &= \int_a^b f(\gamma(t))\gamma'(t)\,dt \\ &= \int_\gamma f(z)\,dz.\end{aligned}$$

□

For this reason, we see that the value of a path integral is dependent on the contour of the path. Let C denote the trace of a piecewise smooth path γ. The value of the integral of an analytic function f on C is independent of the manner in which C is traced out by any particular path. Thus, we may denote the contour integral of f on C as

$$\int_C f(z)\,dz = \int_\gamma f(z)\,dz,$$

where γ is any piecewise smooth path whose image is C.

Example 3. Let L denote the line segment in $\mathbb{C}$ from 0 to $1+i$. Find $\int_L z^2\,dz$.

Solution. Even though the value of the integral is independent of the parameterization, we need a path with which to compute.

If $z_1, z_2 \in \mathbb{C}$, the line segment from z_1 to z_2 may be parameterized by $\gamma : [0,1] \to \mathbb{C}$ given as $\gamma(t) = z_1 + t(z_2 - z_1)$, which is smooth.

Thus, let $\gamma(t) = 0 + t(1+i)$, so that $f(\gamma(t)) = t^2(1+i)^2 = 2it^2$, $\gamma'(t) = 1+i$, and $f(\gamma(t))\gamma'(t) = 2it^2(1+i) = 2(i-1)t^2$. Compute

$$\int_L f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_0^1 2(i-1)t^2\,dt = 2(i-1)\int_0^1 t^2\,dt = -\frac{2}{3} + i\frac{2}{3}.$$

□

5. Negation and Concatenation of Contours

It is important to note that the contour starts at a particular point, and ends at a particular point, for this definition to hold; if we traverse the contour in the opposite direction, we obtain the negative value of original value of the integral.

Let C be a contour, that is, the image of a piecewise continuous path with specified initial and terminal points. Let $-C$ denote the same set, but with the initial and terminal points reversed. Then

$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz.$$

We may *concatenate* to contours C_1 and C_2 if the terminal point of C_1 equals the initial point of C_2. To be completely general and yet precise, let $\gamma_1 : [a_1, b_1] \to \mathbb{C}$ and $\gamma_2 : [a2, b_2] \to \mathbb{C}$ be piecewise continuous, paths with images C_1 and C_2 respectively, such that $\gamma_1(b_1) = \gamma_2(a_2)$. Define a new path

$$\gamma : [0,1] \to \mathbb{C} \quad \text{by} \quad \gamma(t) = \begin{cases} \gamma_1(a_1 + (2t-0)(b_1 - a_1)) & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ \gamma_2(a_2 + (2t-1)(b_2 - a_2)) & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Define the concatenation of C_1 and C_2, denoted by $C_1 + C_2$, to be the contour which is the trace of γ. In this case, it is clear that

$$\int_{C_1+C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$$

By convention, $C_1 - C_2$ mean $C_1 + (-C_2)$.

We now compute a critical example.

Example 4. Let C denote the unit circle with equation $|z| = 1$. We consider C to be a contour, the initial and terminal points being $1 \in \mathbb{C}$.

Let C_1 be the upper half circle, traced from 1 to -1 in the upper half plane.

Let C_2 be the lower half circle, traced from 1 to -1 in the lower half plane.

Then $C = C_1 - C_2$. Let $f(z) = \overline{z}$. Compute the contour integrals of f along C_1, C_2, and C.

Solution. Let $\gamma_1 : [0, \pi] \to \mathbb{C}$ be given by $\gamma_1(t) = e^{it}$; the trace of γ_1 is C_1. Then $f(\gamma_1(t))\gamma_1'(t) = \overline{e^{it}}(ie^{it}) = ie^{-it}e^{it} = i$, so

$$\int_{C_1} f(z)\,dz = \int_0^\pi f(\gamma_1(t))\gamma_1(t)\,dt = \int_0^\pi i\,dt = \pi i.$$

Let $\gamma_2 : [0, \pi] \to \mathbb{C}$ be given by $\gamma_2(t) = e^{-it}$; the trace of γ_2 is C_2. Then $f(\gamma_2(t))\gamma_2'(t) = \overline{e^{-it}}(-ie^{-it}) = -ie^{it}e^{-it} = -i$, so

$$\int_{C_2} f(z)\,dz = \int_0^\pi f(\gamma_2(t))\gamma_2(t)\,dt = \int_0^\pi (-i)\,dt = -\pi i.$$

So, in this case, the value of the integral from 1 to -1 depends on the path we take.

Finally,

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = \pi i - (-\pi i) = 2\pi i.$$

□

We already know that $\overline{z}$ is not an analytic function. However, if $|z| = 1$, then $z\overline{z} = |z| = 1$, so $\overline{z} = \dfrac{1}{z}$. Thus we have shown that

$$\int_C \frac{1}{z}\,dz = 2\pi i.$$

As it turns out, this is the primordial case of nontrivial path integration, as we will experience.

6. Primitives

A primitive is an antiderivative.

Definition 5. Let D be an open connected set and let $f : D \to \mathbb{C}$. A *primitive* for f on D is a function $F : D \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in D$.

Theorem 1. *Let D be an open connected set and let $f : D \to \mathbb{C}$. Suppose that f admits a primitive F in D. Let $z_1, z_2 \in D$. Then for every piecewise smooth path $\gamma : [a, b] \to D$ with $\gamma(a) = z_1$ and $\gamma(b) = z_2$,*

$$\int_\gamma f(z)\,dz = F(z_2) - F(z_1).$$

Proof. Compute

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_a^b f(\gamma(t))\gamma'(t)\,dt \\ &= \int_a^b (F\circ\gamma)'(t)\,dt \\ &= F\circ\gamma(t)\Big|_a^b \\ &= F(\gamma(b)) - F(\gamma(a)) \\ &= F(z_2) - F(z_1).\end{aligned}$$

□

We have previously seen that path integrals do not depend on the parameterization of a given contour. The theorem above state that, under appropriate conditions, the path integral is independent of the path chosen. This is a powerful result, and naturally leads to the following (essentially equivalent) formulation.

A *closed path* is a piecewise smooth path $\gamma : [a, b] \to \mathbb{C}$ such that $\gamma(b) = \gamma(a)$. A *simple closed path* is a closed path which is injective except at the endpoints. Such a path is said to be "in D" if $\gamma(t) \in D$ for all $t \in [a, b]$.

Corollary 1. *Let D be an open connected set and let $f : D \to \mathbb{C}$. Suppose that f admits a primitive F in D. Let γ be a simple closed path in D. Then*

$$\int_\gamma f(z)\,dz = 0.$$

These results lead to several questions:

- Is it possible to weaken the hypothesis of the existence of a primitive?
- Under what conditions does a primitive for f exist in a domain D?
- Is the converse of Theorem 1 true? That is, if integrals of f are path independent in a domain D, does this imply that f has a primitive in D?

7. Construction of Primitives

7.1. **Over the Reals.** We briefly review the construction of antiderivatives in $\mathbb{R}$.

Let $a, b \in \mathbb{R}$ with $a < b$. Let $f : [a, b] \to \mathbb{R}$ be continuous. Define a function

$$F : [a, b] \to \mathbb{R} \quad \text{by} \quad F(x) = \int_a^x f(t)\,dt.$$

We claim that $F'(x) = f(x)$ for all $x \in (a, b)$.

To see this, consider

$$F(x + h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt.$$

Now $\int_x^{x+h} f(t)\,dt$ is the area under the graph of f from x to $x + h$. Since f is continuous, it is clear that, for very small h, this area is approximately the area of the rectangle whose height is $f(x)$ and whose width is h; that is,

$$\int_x^{x+h} f(t)\,dt \approx f(x)h.$$

Thus, for very small h,

$$F'(x) \approx \frac{F(x + h) - F(x)}{h} = \frac{\int_x^{x+h} f(t)\,dt}{h} \approx \frac{f(x)h}{h} = f(x).$$

These approximations become precise as h approaches zero, so

$$F'(x) = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h} = f(x).$$

7.2. **In Star-Shaped Regions.**

Definition 6. Let $D \subset \mathbb{C}$ be open and connected and let $p \in D$. We say that D is a *star-shaped region* with respect to p if, for every $z \in D$, the line segment from p to z is contained in D.

Let $D \subset \mathbb{C}$ be a star-shaped region, and let $f : D \to \mathbb{C}$ be continuous. We would like to mirror the construction of a real antiderivative in this context.

Suppose D is star-shaped with respect to $p \in D$. For each $z \in D$, let L_z denote the line segment from p to z. We know that one parameterization for this line segment is $\gamma_z : [0, 1] \to \mathbb{C}$ given by $\gamma_z(t) = p + (z - p)t$.

Define a function

$$F : D \to \mathbb{C} \quad \text{by} \quad F(z) = \int_{L_z} f(w)\,dw.$$

We would like to show that $F : D \to \mathbb{C}$ is a primitive for f on D. We let h be a small complex number; then L_z and L_{z+h} are line segments starting at p. Let L_* denote the line segment from z to $z + h$.

Following the development over the reals, we consider the numerator of the difference quotient, and we would like to say that

$$F(z + h) - F(z) = \int_{L_{z+h}} f(w)\,dw - \int_{L_z} f(w)\,dw = \int_{L_*} f(w)\,dw.$$

Unfortunately, we do not know that the last equal sign is true. If $z+h$ is on the line through p and z, it certainly is true, but otherwise, we need to assume path independence to write

$$\int_{L_{z+h}} f(w)\,dw = \int_{L_z+L_*} f(w)\,dw = \int_{L_z} f(w)\,dw + \int_{L_*} f(w)\,dw.$$

7.3. **Cauchy-Goursat Theorem.** The Cauchy-Goursat theorem precisely states the conditions under which the conclusion of Corollary 1 is true. It was first shown by Cauchy that if f is differentiable with continuous derivative at every point in the interior of a curve, then the integral of f along that curve is zero. The assumption of continuity of the derivative was later shown to be unnecessary; using this, one can show that differentiability in an open set implies continuity of the derivative in that open set.

There are many formal proofs of various versions of the Cauchy-Goursat Theorem available in excellent books such as Ahlfors' or Conway's graduate level texts. The approach is either to reference Green's Theorem from Vector Calculus, or to subdivide the domain into small pieces. Churchill and Ahlfors both begin by breaking up a rectangle into smaller rectangles.

We give an informal explanation using the latter approach, and restate the theorem in a slightly different context. One of the more intuitive forms of the statement of this theorem expresses that a continuous deformation of a path though a region of analyticity does not change the value of the path integral. The notion of continuous deformation is made precise with the definition of homotopy.

8. Homotopy

Loosely speaking, given two paths, a homotopy from the first to the second is a continuous deformation. Imagine one path moving through the plane until it lies on top of the other. We want the initial point of the first to move to the initial point of the second, and the terminal point of one to move to the terminal point of the other. That is, we want a family of paths γ_t such that γ_0 is one path, γ_1 is the other path, and as t increases, γ_0 moves until it is the same as γ_1.

We wish to define the notion of "continuous deformation of a path" rigorously. So, for this section, to conform with the common way to writing this, we need to slightly change our notation. Let $I=[0,1]$ denote the closed unit interval. Without loss of generality, we may assume that each path we consider has I as a domain. This is because, if we start with $\gamma : I \to D$, we may reparameterize it to be $\widehat{\gamma} : I \to D$ by $\widehat{\gamma}(s) = \gamma(a+t(b-a))$. Then γ and $\widehat{\gamma}$ have the same trace, and produce the same values for any function integrating along them. Thus we will use I as the domain for our paths, and we will use s as a parameter for I. For example, $\gamma(s) = 1 + (i-1)s$ is a path from 1 to i.

Definition 7. Let D be a subset of $\mathbb{C}$. Let $\gamma_0 : I \to D$ and $\gamma_1 : I \to D$ be paths in D. A *homotopy in* D from γ_0 to γ_1 is a continuous function

$$\Gamma : I \times I \to D$$

such that

- $\Gamma(s,0) = \gamma_0(s)$ for all $s \in I$
- $\Gamma(s,1) = \gamma_1(s)$ for all $s \in I$

Let $\gamma_t(s) = \Gamma(s,t)$. Then $\gamma_t : I \to D$ is a path in D. The family $\{\gamma_t \mid t \in I\}$ is a referred to as a continuous deformation from γ_0 to γ_1.

We say that γ_0 is *homotopic* to γ_1 if there exists a homotopy from γ_0 to γ_1.

It is fairly easy to see that if D is connected, then any two paths are homotopic. We can just shrink γ_0 to a point, move the point, and then expand to γ_1. For our purposes, we need the concept of fixed endpoint homotopy.

Definition 8. Let D be a subset of $\mathbb{C}$. Let $u, v \in D$. Let $\alpha : I \to D$ and $\beta : I \to D$ be paths in D from u to v. That is, $u = \alpha(0) = \beta(0)$ and $v = \alpha(1) = \beta(1)$. A *fixed endpoint homotopy in D* from α to β is a continuous function

$$\Gamma : I \times I \to D, \qquad \text{where} \qquad \gamma_t(s) = \Gamma(s,t),$$

such that

- $\gamma_0(s) = \alpha(s)$ for all $s \in I$
- $\gamma_1(s) = \beta(s)$ for all $s \in I$
- $\gamma_t(0) = u$ for all $t \in I$
- $\gamma_t(1) = v$ for all $t \in I$

We say that α and β are *homotopic in D*, or *homotopy equivalent in D*, if there exists a fixed endpoint homotopy in D from α to β.

Henceforth, we assume that all homotopies of paths are fixed endpoint homotopies.

Definition 9. Let $D \subset \mathbb{C}$. A *constant path* in D is a path $\gamma : [0,1] \to D$ such that $\gamma(s_1) = \gamma(s_2)$ for all $s_1, s_2 \in [0,1]$. That is, if $\gamma(0) = u$, then $\gamma(s) = u$ for all $s \in [0,1]$.

Definition 10. Let $D \subset \mathbb{C}$. A *loop* in D is a continuous function $\gamma : [0,1] \to D$ such that $\gamma(a) = \gamma(b)$. The loop is *simple* if

$$\gamma(s_1) = \gamma(s_2) \quad \Rightarrow \quad s_1 = s_2 \text{ or } s_1, s_2 \in \{a, b\}.$$

Definition 11. Let $D \subset \mathbb{C}$. Let $\gamma : [0,1] \to D$ be a loop in D. We say that γ is *contractible* if γ is homotopic to a constant in D.

Definition 12. Let $D \subset \mathbb{C}$. We say that D is *simply connected* if every loop in D is contractible.

Definition 13. Let $\alpha : I \to \mathbb{C}$ and $\beta : I \to \mathbb{C}$ be paths in D. The *concatenation* of α and β is a new path

$$\alpha * \beta : [0,1] \to \mathbb{C} \quad \text{by} \quad \alpha * \beta(s) = \begin{cases} \alpha(2s) & \text{if } s \in \left[0, \frac{1}{2}\right]; \\ \beta(2s-1) & \text{if } s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

The *inverse* of α is a new path

$$\alpha^{-1} : [0,1] \to \mathbb{C} \quad \text{by} \quad \alpha^{-1}(s) = \alpha(1-s).$$

9. The Cauchy-Goursat Theorem

Theorem 2. *Let D be an open subset of $\mathbb{C}$. Let $f : D \to \mathbb{C}$ be analytic on D. Let α and β be homotopic paths in D. Then*

$$\int_\alpha f(z)\,dz = \int_\beta f(z)\,dz.$$

Corollary 2. *Let D be an open, connected, simply connected subset of $\mathbb{C}$. Let f be analytic on D. Then for every simple closed curve C in D,*

$$\int_C f(z)\,dz = 0.$$

We review the conditions in the hypothesis of the Cauchy-Goursat Theorem.

- D is open if every point in D has a neighborhood which is contained in D.
- D is connected if there is a path in D between any two points in D.
- D is simply connected if every loop in D is contractible.
- f is analytic on D if f is differentiable at every point in D.
- C is a simple closed curve if it is the image of a path which is injective except that the initial point equals the terminal point.

We begin by discussing why the theorem and the corollary are equivalent. Suppose that α and β are paths in D from z_1 to z_2. Then $\gamma = \alpha * \beta^{-1}$ is a loop based at z_1. We have

$$\int_\gamma f(z)\,dz = \int_\alpha f(z)\,dz - \int_\beta f(z)\,dz.$$

So the integral along γ is zero if and only if the integrals along α and β are equal.

Let D be an open subset of $\mathbb{C}$. Let $f : D \to \mathbb{C}$ be differentiable at every point in D. Let α and β be homotopic paths in D. Set $\gamma = \alpha * \beta^{-1}$. We wish to see why the integral of f along γ is zero.

Let $\Gamma : I \times I \to D$ be a homotopy from α to β. Let n be a positive integer. We break up the square $I \times I$ into n^2 subsquares as follows. Set $s_i = \dfrac{i}{n}$ and $t_j = \dfrac{j}{n}$, so that $\{s_i\}$ is a partition of the first copy of I, and $\{t_j\}$ is a partition of the second. Let r_{ij} denote the path along the subsquare from (s_{i-1}, t_{j-1}), to (s_{i-1}, t_j), to (s_i, t_j), to (s_i, t_{j-1}), and back to (s_{i-1}, t_{j-1}). Let $\gamma_{ij} = \Gamma(r_{ij})$. Then γ_{ij} is a closed loop in D.

The first key idea is that

$$\int_\gamma f(z)\,dz = \sum_{i,j} \int_{\gamma_{ij}} f(z)\,dz.$$

The reason is that, for each edge of one of the squares, the integral along the image of that edge is traversed in both directions, *unless that edge is on part of the boundary of $I \times I$*. Thus the integrals for the interior line segments all cancel, and we are left with only the integrals along the boundary of $I \times I$.

Let z_{ij} denote the image under Γ of the center of each of the subsquares; that is, $z_{ij} = \Gamma(\frac{s_{i-1}+s_i}{2}, \frac{t_{j-1}+t_j}{2})$. Then f is differentiable at z_{ij}. Thus, if we let n be large enough,

$$|\frac{f(z)-f(z_{ij})}{z-z_{ij}} - f'(z_{ij})| < \epsilon,$$

for whatever ϵ we choose. For very small ϵ, we may write

$$\frac{f(z)-f(z_{ij})}{z-z_{ij}} \approx f'(z_{ij}).$$

Solving this for $f(z)$ produces

$$f(z) \approx c_1 z + c_2 \quad \text{where} \quad c_1 = f'(z_{ij}) \text{ and } c_2 = f(z_{ij}) - f'(z_{ij})z_{ij}.$$

This approximation should be very good near z_{ij}, so assuming γ_{ij} fits within a disk of radius ϵ around z_{ij}, we have

$$\int_{\gamma_{ij}} f(z)\,dz \approx \int_{\gamma_{ij}} c_1 z\,dz + \int_{\gamma_{ij}} c_2\,dz.$$

Since c_1 and c_2 are constant,

$$\int_{\gamma_{ij}} f(z)\,dz \approx c_1 \int_{\gamma_{ij}} z\,dz + c_2 \int_{\gamma_{ij}} dz.$$

We have previous established that the last two integrals are zero, so

$$\int_{\gamma_{ij}} f(z)\,dz \approx 0.$$

This approximation becomes precise as $n \to \infty$. Thus,

$$\int_{\gamma} f(z)\,dz = \sum_{ij} \int_{\gamma_{ij}} f(z)\,dz = \sum_{ij} 0 = 0.$$

10. Exercises

Problem 1. Let $a = 0$ and $b = 1 + i$. Let γ be the path from a to b given by $\gamma(t) = a + t(b - a)$ for $t \in [0, 1]$. Use Definition 4 to compute the following integrals.

(a) Let $f(z) = 1$. Compute $\int_\gamma f(z)\,dz$.
(b) Let $f(z) = z$. Compute $\int_\gamma f(z)\,dz$.
(c) Let $f(z) = z^2$. Compute $\int_\gamma f(z)\,dz$.

Problem 2. Let $a = 0$ and $b = 1+i$. Let γ be the path from a to b given by $\gamma(t) = t+it^2$ for $t \in [0, 1]$. Use Definition 4 to compute the following integrals.

(a) Let $f(z) = 1$. Compute $\int_\gamma f(z)\,dz$.
(b) Let $f(z) = z$. Compute $\int_\gamma f(z)\,dz$.
(c) Let $f(z) = z^2$. Compute $\int_\gamma f(z)\,dz$.

Problem 3. Let $a \in \mathbb{C}$ and let γ parameterize the circle of radius 1 about 0 by $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$. Use Definition 4 to compute the following integrals.

(a) Let $f(z) = 1$. Compute $\int_\gamma f(z)\,dz$.
(b) Let $f(z) = z$. Compute $\int_\gamma f(z)\,dz$.
(c) Let $f(z) = \frac{1}{z}$. Compute $\int_\gamma f(z)\,dz$.

Problem 4. You are given a function $f(z)$ and a pair of points $z_1, z_2 \in \mathbb{C}$. Determine if the hypothesis of Theorem 1 holds. If so, specify D, and use the theorem to compute $\int_\gamma f(z)\,dz$, where γ is any path in D from z_1 to z_2. If not, describe why not, find a path γ from z_1 to z_2, and compute $\int_\gamma f(z)\,dz$.

(a) $f(z) = \cos(z)$, $z_1 = 1$, $z_2 = i$.
(b) $f(z) = z^2$, $z_1 = 1$, $z_2 = i$.
(c) $f(z) = \frac{1}{z}$, $z_1 = 1$, $z_2 = i$.
(d) $f(z) = \frac{1}{z^2}$, $z_1 = 1$, $z_2 = i$.

Problem 5. You are given a function $f(z)$, a point z_0, and a radius r. Let C be a circle of radius r about z_0. Determine if f has a primitive in an open set containing the disk of radius r about z_0. If so, we know that $\int_C f(z)\,dz = 0$. If not, compute $\int_C f(z)\,dz$.

(a) $f(z) = \cos(z)$, $z_0 = i$, $r = 1$.
(b) $f(z) = \frac{1}{z}$, $z_0 = 2$, $r = 1$.
(c) $f(z) = \frac{1}{z}$, $z_0 = 2$, $r = 3$.
(d) $f(z) = \frac{1}{z^2}$, $z_0 = 2$, $r = 3$.

Problem 6. Let $f(z) = z^2$ and let $\gamma : [0, 1] \to \mathbb{C}$ be given by $\gamma(t) = 1 + t(i - 1)$. Use Definition 4 to compute $\int_\gamma f(z)\,dz$.

Problem 7. Let $f(z) = \overline{z}$ and let C be a circle of radius 1 about the origin. Use Definition 4 to compute $\int_C f(z)\,dz$.

Problem 8. Let $f(z) = z^3 - z$ and let γ be any path from 2 to $2i$. Use Theorem 1 to compute $\int_\gamma f(z)\,dz$.

Problem 9. Let $f(z) = \frac{1}{z}$ and let C be the line segment from $\sqrt{e}$ to $\sqrt{3}+i$. Use Theorem 1 to compute $\int_\gamma f(z)\,dz$.

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